Coefficient Bounds for p-valent Functions Associated with Quasi-Subordination

Mohammed H. Saloomi  
Department of Mathematics  
University of Karbala  
Karbala, Iraq

Abbas Kareem Wanas  
Department of Mathematics  
University of Al-Qadisiyah  
Diwaniya, Iraq

Enaam Hadi Abd  
Department of Computer Sciences,  
University of Kerbala,  
Karbala, Iraq

Abstract- In this paper, the upper bounds of $|a_{p+1}|$ and $|a_{p+2}|$ are determined for new certain subclasses of p-valent holomorphic functions defined by quasi-subordination. Also, we obtain the Fekete-Szegö inequalities for the functions belonging to these subclasses.

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MSC: 30C45; 30C80.

1. INTRODUCTION

Let us denote by $A_p$ the class of all functions $f$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n.$$  \hfill (1.1)

which are holomorphic in the open unit disk $\mathcal{U}$.

A holomorphic function $f(z)$ is subordinate to another holomorphic function $H(z)$ if there exist holomorphic functions $\omega(z)$ with $\omega(0) = 0$, $|\omega(z)| < 1$ in $\mathcal{U}$, such that

$$f(z) = H(\omega(z)).$$

We denote this subordination as following

$$f < H, \text{ or } f(z) < H(z)(z \in \mathcal{U}). \hfill (1.2)$$

Also, if the function $H$ is univalent in $\mathcal{U}$, then $f(z) < H(z)$ is equivalent to $f(0) = H(0)$ and $f(\mathcal{U}) \subset H(\mathcal{U})$. For more details on the concept of subordination, see [1].

Robertson [2] introduced the concept of quasi-subordinate. For two holomorphic functions $f$ and $H$, the function $f$ is quasi-subordination to $H$, written as follows:

$$f(z) \preceq_H H(z), (z \in \mathcal{U}) \hfill (1.3)$$

if there exist holomorphic functions $\varphi$ and $\omega$ with $|\varphi(z)| \leq 1$, $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$f(z) = \varphi(z) H(\omega(z)), \quad (z \in \mathcal{U}).$$

Observe that when $\varphi(z) = 1$, then $f(z) = H(\omega(z))$ so that $f(z) < H(z)$ in $\mathcal{U}$. Also notice that, if $\omega(z) = z$, then $f(z) = \varphi(z) H(z)$ and it is said that $f$ is majorized by $H$, in $\mathcal{U}$. Hence it is obvious that
quasi-subordination is generalization of subordination as well as majorization. Joining the concepts of subordination and majorization (see [3]). Also for works related to quasi-subordination (see [2]).

In this investigation we assume that
\[
\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \cdots,
\]
\[
\phi(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots
\]  \hspace{1cm} (1.4)
and
\[
\psi(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots, \quad b_1 > 0
\]  \hspace{1cm} (1.5)
Now we shall define the following subclasses of \(A_p\):

**Definition (1.1):** A function \(f \in A_p\) is said to be in the class \(M^p_\lambda(\lambda, \alpha, \psi)(\lambda \in \mathbb{C} - \{0\}, p \in \mathbb{N}, 0 \leq \alpha \leq 1\) if it satisfies the following quasi-subordination condition
\[
(1 - \alpha) \left[ \frac{1}{\lambda}(\frac{zf'(z)}{f(z)} + 1) \right] + \alpha \left[ \frac{zf''(z)}{f(z)} - (p - 1) \right] < q \psi(z) - 1.
\]

**Definition (1.2):** A function \(f \in A_p\) is said to be in the class \(\mathcal{H}^p_\lambda(\lambda, \alpha, \psi)(\lambda \in \mathbb{C} - \{0\}, \alpha \leq 1, p \in \mathbb{N})\), if it satisfies the following quasi-subordination condition
\[
\frac{1}{\lambda}(\frac{zf'(z)}{f(z)} - 1) + \frac{\alpha}{\lambda}(\frac{f''(z)}{zf'(z)} - 1) < q \psi(z) - 1
\]
In this paper we shall get the Fekete-Szego inequalities for \(M^p_\lambda(\lambda, \alpha, \psi)\) and \(\mathcal{H}^p_\lambda(\lambda, \alpha, \psi)\) as well as its special classes. Several authors have also investigated the bounds for the Fekete-Szego coefficient for various classes. More details for Fekete-Szego problem to certain related classes of functions see ([1, 2, 3, 4, 5, 6, 7, 8]). Recently, Saloomi and many authors have examined bounds for different subclasses of P-valent functions [9]. With a view to derive our main results, we have to recall here the following lemmas.

**Lemma (1.3):**[10] If \(\omega\) be the function holomorphic in the disk \(\mathcal{U}\) satisfying \(\omega(0) = 0\) and \(|\omega(z)| < 1\) and let \(\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \cdots\). Then and for any complex number \(S\),
\[
|\omega_2 - S\omega_1|^2 \leq \max\{1, |S|\}.
\]

**Lemma (1.4):**[10] If \(\phi\) be holomorphic function in the open unit disk \(\mathcal{U}\) satisfying \(|\phi(z)| < 1\), and let \(\phi(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots\). Then \(|c_0| \leq 1\), and
\[
|c_1| \leq 1 - |c_0|^2.
\]

### 2. Main Results

**Theorem (2.1).** If \(f \in A_p\) is given by (1.1) belong to \(M^p_\lambda(\lambda, \alpha, \psi)\), then
\[
|a_{p+1}| \leq \frac{|\lambda|b_1 p^2}{2(p + 1)(1 - \alpha + ap)} \left[ 1 + \max \left\{ 1, \frac{|\lambda|b_1 p^2 (p + 1)^2 (1 - \alpha + ap)}{[(1 - \alpha)(p + 1) + ap^2 (p + 1)]^2} + \frac{|b_2|}{|b_1|} \right\} \right],
\]  \hspace{1cm} (2.1)
and
\[
|a_{p+2} - \mu a_{p+1}^2| \leq \frac{|\lambda|b_1 p}{2(p + 2)(1 - \alpha + ap)} \left[ 1 + \max \left\{ 2\mu |\lambda|b_1 p^2 (p + 2) (1 - \alpha + ap), \frac{|\lambda|b_1 p^2 (p + 1)^2 (1 - \alpha + ap)}{[(1 - \alpha)(p + 1) + ap^2 (p + 1)]^2} + \frac{|b_2|}{|b_1|} \right\} \right] \hspace{1cm} (2.2)
\]
Proof: If \( f(z) \in \mathcal{M}^p(\lambda, \alpha, \psi) \), then there are holomorphic functions in \( \mathcal{U} \) say \( \omega \) and \( \phi \) with \( \omega(0) = 0 \), \( |\omega(z)| < 1 \) and \( |\phi(z)| \leq 1 \), such that:

\[
(1 - \alpha) \left[ \frac{z f^{(n)}(z)}{f'(z)} + 1 \right] - \frac{1}{\lambda} + \frac{\alpha z f''(z)}{f'(z)} - (p - 1) = \phi(z)(\psi(\omega(z)) - 1) \quad (2.3)
\]

Since

\[
(1 - \alpha) \left[ \frac{1}{\lambda} \frac{z f''(z)}{f'(z)} + 1 \right] - \frac{1}{\lambda} + \frac{\alpha z f''(z)}{f'(z)} - (p - 1) = \left[ \frac{(1 - \alpha)(p + 1) + \alpha(p + 1)}{\lambda p^2} \right] a_{p+1} + \left( \frac{1 - \alpha + \alpha p}{\lambda p} \right) 2(p + 2)a_{p+2} - \frac{(p + 1)^2}{p} a_{p+1}^2 \right] z^2 + \ldots. \quad (2.4)
\]

Also

\[
\phi(z)(\psi(\omega(z)) - 1) = c \cdot b_1 \omega_1 z + [c_1 b_1 \omega_1 + c_0 (b_1 \omega_2 + b_2 \omega_1^2)] z^2 \ldots \quad (2.5)
\]

Putting (2.4) and (2.5) in (2.3) and equating coefficient both sides, we have

\[
\left( \frac{(1 - \alpha)(p + 1) + \alpha(p + 1)}{\lambda p^2} \right) a_{p+1} = c_0 b_1 \omega_1,
\]

applying Lemma(1.3) and Lemma(1.4), we get

\[
|a_{p+1}| \leq \frac{|\lambda| b_1 p^2}{(1 - \alpha)(p + 1) + \alpha p^2(p + 1)}.
\]

Moreover, we have

\[
1 - \alpha + \alpha p \left( \frac{2(p + 2)a_{p+2} - \frac{(p + 1)^2}{p} a_{p+1}^2}{\lambda p} \right) = [c_1 b_1 \omega_1 + c_0 (b_1 \omega_2 + b_2 \omega_1^2)],
\]

or equivalently

\[
a_{p+2} = \frac{\lambda p}{2(p + 2)(1 - \alpha + \alpha p)} [c_1 b_1 \omega_1 + c_0 (b_1 \omega_2 + b_2 \omega_1^2)] + \frac{\lambda^2 p^3 c_0^2 \omega_1^2 b_1^2 (p + 1)^2}{2(p + 2)(1 - \alpha + \alpha p^2)^2}.
\]

Also

\[
a_{p+2} - \mu a_{p+1}^2 = \frac{\lambda p}{2(p + 2)(1 - \alpha + \alpha p)} [c_1 b_1 \omega_1 + c_0 (b_1 \omega_2 + b_2 \omega_1^2)] + \frac{\lambda^2 p^3 c_0^2 \omega_1^2 b_1^2 (p + 1)^2}{2(p + 2)(1 - \alpha + \alpha p^2)^2} - \mu \frac{\lambda^2 p^4 c_0^2 b_1^2 \omega_1^2}{(p + 1)(1 - \alpha + \alpha p^2)^2}.
\]

Applying, Lemma(1.3) and Lemma(1.4), on previous relation, we get the result (2.2).

The case \( \mu = 0 \), gives the estimate of \( |a_{p+2}| \). The proof is complete.

Remark (2.2): For \( \alpha = 1 \) in the class \( \mathcal{M}^p(\lambda, \alpha, \psi) \) then we have the class \( \mathcal{C}^p(\lambda, \psi) \) was introduced in [11]. For \( \alpha = 1, \) \( p = 1 \) in the class the class \( \mathcal{M}^p(\lambda, \alpha, \psi) \) then we have the class \( \mathcal{C}^p(\lambda, \psi) \) was introduced in [11], for the case when \( \alpha = 1, \) \( p = 1 \) and \( \lambda = 1 \) in the class \( \mathcal{M}^p(\lambda, \alpha, \psi) \) reduce to the class \( \mathcal{C}^p(\psi) \) [12].

Remark (2.3): For \( \phi(z) = 1 \) and \( \lambda = 1 \), we have \( \mathcal{M}^p(\alpha, \psi) \) in the following corollary:

Corollary (2.4): If \( f \) be in the class \( \mathcal{M}^p(\alpha, \psi) \), then

\[
|a_{p+1}| \leq \frac{|\lambda| b_1 p^2}{(1 - \alpha)(p + 1) + \alpha p^2(p + 1)}
\]

and

\[
|a_{p+2}| \leq \frac{1}{2(p + 1)(1 - \alpha + \alpha p)} \max \left\{ \frac{b_2}{b_1}, \frac{b_1 b_2^2(p + 1)^2(1 - \alpha + \alpha p)}{(1 - \alpha)(p + 1) + \alpha p^2(p + 1)^2} \right\} + \frac{b_2}{b_1}.
\]
\[|\alpha_{p+2} - \mu\alpha_{p+1}^2| \leq \frac{b_1 p}{2(p + 2)(1 - \alpha + \alpha p)} \max \left( \frac{2|\lambda|b_1 p^3(1 - \alpha + \alpha p)}{((1 - \alpha)(p + 1) + \alpha^2(p + 1))^2} + \frac{b_2}{b_1} \right) \]

**Theorem (2.5).** If \( f \in A_p \) satisfies
\[
(1 - \alpha) \left( \frac{1}{\lambda p} \left( \frac{zf''(z)}{f'(z)} + 1 \right) - \frac{1}{\lambda} \right) + \frac{zf''(z)}{f'(z)} - (p - 1) \leq \psi(z) - 1.
\]
then the following inequalities hold:
\[
|\alpha_{p+1}| \leq \frac{b_1 p^2}{(1 - \alpha)(p + 1) + \alpha^2(p + 1)}
\]
\[
|\alpha_{p+2}| \leq \frac{|\lambda|b_1 p}{2(p + 2)(1 - \alpha + \alpha p)} \left[ 1 + \frac{|\lambda|b_1 p^2(1 - \alpha + \alpha p)}{[1 - \alpha + \alpha^2p^2]^2} + \frac{b_2}{b_1} \right].
\]
and
\[
|\alpha_{p+2} - \mu\alpha_{p+1}^2| \leq \frac{|\lambda|b_1 p}{2(p + 2)(1 - \alpha + \alpha p)} \left[ 1 + \frac{|\lambda|b_1 p^2(1 - \alpha + \alpha p)((p + 1)^2 - 2\mu p(p + 2))}{(p + 1)^2[1 - \alpha + \alpha^2p^2]^2} + \frac{b_2}{b_1} \right].
\]

**Proof:** The proof is obtained as in Theorem (2.1), setting \( \omega(z) = z \).

**Remark:** If \( \alpha = 1 \), we have Theorem (2.5) in [13], if \( \alpha = 0 \) in Theorem (2.5), then we have the following:

**Corollary (2.6):** If \( f \in A_p \) satisfies
\[
\left( \frac{1}{\lambda p} \left( \frac{zf''(z)}{f'(z)} + 1 \right) - \frac{1}{\lambda} \right) \leq \psi(z) - 1.
\]
then the following inequalities hold:
\[
|\alpha_{p+1}| \leq \frac{b_1 p^2}{(p + 1)}
\]
\[
|\alpha_{p+2}| \leq \frac{|\lambda|b_1 p}{2(p + 1)} \left[ 1 + \frac{|\lambda|B_1 p^2}{b_1} \right].
\]
and
\[
|\alpha_{p+2} - \mu\alpha_{p+1}^2| \leq \frac{|\lambda|b_1 p}{2(p + 2)(1 - \alpha + \alpha p)} \left[ 1 + \frac{|\lambda|b_1 p^2((p + 1)^2 - 2\mu p(p + 2))}{(p + 1)^2} + \frac{b_2}{b_1} \right].
\]

**Theorem (2.7):** If \( f \in A_p \) is given by (1.1) belong to \( \mathcal{H}_q^P(\lambda, \alpha, \psi) \), then
\[
|\alpha_{p+1}| \leq \frac{|\lambda|b_1 p_{\lambda p}}{\alpha(p + 1) + 1} \quad (2.6)
\]
\[
|\alpha_{p+2}| \leq \frac{|\lambda|b_1 p_{\lambda p}}{2\alpha(p + 2) + 1} \left[ 1 + \max \left( 1, \frac{|\lambda|b_1 p_{\lambda p}}{[\alpha(p + 1) + 1]} + \frac{b_2}{b_1} \right) \right]. \quad (2.7)
\]
\[
|\alpha_{p+2} - \mu\alpha_{p+1}^2| \leq \frac{|\lambda|b_1 p_{\lambda p}}{2\alpha(p + 2) + 1} \left[ 1 + \max \left( 1, \frac{|\lambda|b_1 p_{\lambda p}2\alpha(p + 2) + 1]}{\alpha(p + 1) + 1} + \frac{b_2}{b_1} \right) \right]. \quad (2.8)
\]

**Proof:** If \( f(z) \in \mathcal{H}_q^P(\lambda, \alpha, \psi) \), then there is holomorphic functions in \( \mathcal{U} \) say \( \omega \) and \( \phi \) with \( \omega(0) = 0 \), \( |\omega(z)| < 1 \) and \( |\psi(z)| \leq 1 \), such that:
\[
\left( \frac{zf''(z)}{\lambda f(z)} - 1 \right) + \frac{zf''(z)}{\lambda (p+1)} - 1 = \phi(z)\psi(\omega(z)) - 1. \quad (2.9)
\]
Since
1 \left( \frac{f''(x)}{p f'(x)} - 1 \right) + \frac{a}{\lambda} \left( \frac{f'(x)}{p x^{p-1}} - 1 \right) = \frac{a(p+1)+1}{\lambda p} a_{p+1}^2 + \left( \frac{2a(p+2)+1}{\lambda p} a_{p+2} - \frac{1}{\lambda p} a_{p+1}^2 \right) x^2 + \cdots. \quad (2.10)

Putting (2.10) and (2.5) in (2.9) and equating coefficient both sides, we have

\[
\frac{a(p+1)+1}{\lambda p} a_{p+1}^2 = c \cdot b_1 \omega_1.
\]

Moreover, we have

\[
a_{p+2} = \left[ c_1 \omega_1 + c_2 \left( \frac{b_2}{a(p+1)+1} \right) \omega_1^2 \right].
\]

Also

\[
a_{p+2} - \mu a_{p+1}^2 = \frac{a}{2[a(p+2)+1]} \left[ b_1 c_1 \omega_1 + b_1 c_2 \left( w_2 + \frac{b_2}{a(p+1)+1} \right) \omega_1^2 \right]
\]

Applying Lemma(1.3) and Lemma(1.4), we get (2.6).

Put \( \mu = 0 \), we obtain the estimate of \( |a_{p+2}| \). In (2.7). The proof is complete.

**Remark(2.8):** For \( \alpha = 0 \) and \( \lambda = 1 \) in previous theorem then, we have the class in [13] and if \( \alpha = 0, \lambda = 1, p = 1 \) and \( \phi(x) = 1 \), in previous theorem, we have the class in [14].

For \( \alpha = 1 \) and \( \phi(x) = 1 \), we have the class \( \mathcal{H}_p^x(\lambda, \psi) \).

**Corollary (2.9):** Let \( f \) be in the class \( \mathcal{H}_p^x(\lambda, \psi) \). Then

\[
|a_{p+1}| \leq |\lambda| b_1 p,
\]

\[
|a_{p+2}| \leq \frac{|\lambda| p b_1}{2} \max \left\{ 1, |\lambda| b_1 p + \left| \frac{b_2}{b_1} \right| \right\}
\]

and

\[
|a_{p+2} - \mu a_{p+1}^2| \leq \frac{|\lambda| p b_1}{2} \max \left\{ 1, |\lambda| b_1 p |\mu| + 1 + \left| \frac{b_2}{b_1} \right| \right\}
\]

**Theorem (2.10).** If \( f \in A_p \) satisfies

\[
\frac{1}{\lambda} \left( \frac{f''(x)}{p f'(x)} - 1 \right) + \frac{a}{\lambda} \left( \frac{f'(x)}{p x^{p-1}} - 1 \right) \ll \psi(x) - 1.
\]

then the following inequalities hold:

\[
|a_{p+1}| \leq \frac{|\lambda| b_1 p}{a(p+1)+1}
\]

\[
|a_{p+2}| \leq \frac{|\lambda| p b_1}{2(a(p+2)+1)} \left[ 1 + \frac{|\lambda| b_1 p}{a(p+1)+1} + \left| \frac{b_2}{b_1} \right| \right]
\]

\[
|a_{p+2} - \mu a_{p+1}^2| \leq \frac{|\lambda| b_1 p}{2(a(p+2)+1)} \left[ 1 + \frac{|\lambda| b_1 p}{a(p+1)+1} + \frac{1}{a(p+1)+1} \right] + \frac{|\lambda| b_1 p}{a(p+1)+1} + \left| \frac{b_2}{b_1} \right|
\]

**Proof:** The proof is obtained as in Theorem (2.7), setting \( \omega(x) = x \).

**Remark(2.11):** If \( \alpha = 0 \) and \( \lambda = 1 \), we have Theorem (2.3) in [15].
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