ABSTRACT: We study the analysis of a single server batch arrival retrial queue with optional re-service under modified Bernoulli vacation which consists of a breakdown and delay time. The main assumption in this paper is that the repair process does not start immediately after a breakdown and there is a delay time waiting for repairs to start. We also assume that the customers that the customers who find the server busy are queued in the orbit in accordance with an FCFS discipline. Once a service is completed the customer may leave the system or some of them have the option to demand re-service for the same service taken. At the completion epoch of each service, if there are no customers in the system, the server will wait for the next customer to arrive with probability 1- a and chooses vacation with probability a. The steady state have been found by using supplementary variable technique and compared with known results.

KEY WORDS: Batch arrival, Retrial queue, optional re-service, modified Bernoulli vacation, breakdowns

I. INTRODUCTION

In recent years, the retrial queueing system has been studied extensively, due to its applicability in telephone switching system, telecommunication networks and computer networks. Extensive surveys of retrial queues can be seen in Artalejo (1999) and Gomez – Corral (2008). A review of the literature on retrial queues and their applications can be found in Falin (1997) and Yang and Templeton (1987). Most of the recent studies have been devoted to batch arrival vacation models under different vacation policies. A batch arrival queue with single vacation policy is discussed by Choudhury (2002).

We can find limited studies on queuing system with re-service, researchers like Rajadurai et al. (2014) and Radha et al. (2017). Re-service is also an important factor in the theory of queuing system and have many applications in the real world. There may be situations where re-service is desired, for instance while visiting a doctor, the patient may be recommended for some investigations, after which he may need to see the doctor again while in some situations offering public service a customer may find his service unsatisfactory and consequently demand re-service. In fact, authors like Madan et al. (2004), Jeyakumar and Arumuganathan (2011) studied queues with re-service.

Choudhury and Deka (2008) have analyzed the M/G/1 retrial queue with server subjected to repairs and breakdowns. A number of papers (Krishnakumar and Arivudainambhi (2002), Krishnakumar and PavaiMadheswari (2003) have recently appeared in the queueing literature in which the concept of general retrial time has been considered along with the Bernoulli vacation schedule. However in the retrial queuing systems the server may not be aware of the status of the customers in the system as in the case of Keilson and Servi’smodel (1986). Hence it is
necessary to modify the Bernoulli schedule introduced by them in the retrial context. This special kind of Bernoulli vacation is called as modified Bernoulli vacation schedule considered by Madan (2004), PavaiMadheswari, Krishnakumar and Suganthi (2017). Wang and Li (2008) have studied a similar model with breakdown and repair.

Here we have discussed the case of optional re-service. In this case a customer has the choice of selecting any one of the two types of regular and optional re-service, subsequently has the option to repeat the service taken by him or leaves from the system. We assume customers arriving in batches and the system is providing in regular and optional re-service, the customer can choose re-service for the same service taken [17]. We can find many real-life applications of this model. Such situations are mostly observed in telephone consultation of medical service systems and in the area of computer processing systems, banks, post offices etc. where the customer has option of choosing re-service [14].

The rest of this paper is organized as follows: the description of the mathematical model is given in section 2. In section 3, we introduce the definitions and notations of this chapter. In section 4, all the steady state equations governing the mathematical system of our model formulated, while in section 5, to obtain steady state results in terms of the probability generating functions for the number of customers in the queue, the average number of customers & the average waiting time in the queue. Some special cases are given in section 6 to show the relations between this work and previous works done by other researchers. Conclusion of this work is presented in section 7.

II. THE MATHEMATICAL MODEL

The detailed description of the model is given as follows:

1. We consider an M/G/1 queueing system, according to an arrival of a job follows a compound Poisson process with \( \lambda \). Let \( X \) denotes the number of customers belonging to the \( j \)th arrival batch where \( X_i, i = 1, 2, 3, \ldots \). \( P[X_i = n] = C_i, n = 1, 2, 3, \ldots \) and \( X(z) \) denotes the probability generating function of \( X \).

2. If there is no waiting space and therefore if an arriving customer finds the server being busy to serve his request immediately, all these customers leave the service area and join a pool of blocked customers called orbit. The customers in the orbit, later on try to get their service. Successive Inter retrial times of any customer has an arbitrary probability distribution \( A(x) \) with the corresponding density function \( a(x) \) and Laplace-Stieljies’s transform (LST) \( A^*(x) \). The conditional completion rates for retrial time is

\[
a(x)dx = \frac{dA(x)}{1-A(x)}
\]

3. A single server provides regular and optional re-service to each job. The service discipline is FCFS (First come First served). As soon as regular service is completed may opt for optional re-service for the same service taken without joining the orbit with probability \( r \) or may leave the system with probability \( 1-r \). It is assumed that the service time follows general random variable \( B_1 \) and \( B_2 \) with distribution function \( B_1(t) \) for normal service and \( B_2(t) \) for optional re-service and LST \( B_1^*(x), B_2^*(x) \). The conditional completion rates for service is

\[
\mu_1(x)dx = \frac{dB_1(x)}{1-B_1(x)}, \quad \mu_2(x)dx = \frac{dB_2(x)}{1-B_2(x)}
\]

4. After completion of service to each customer, the server may go for a vacation of random length \( V \) with probability \( a \) (0 \( \leq a \leq 1 \)) or may wait in the service facility to serve the next customer with probability 1 – \( a \). It is assumed that when there are no customers waiting in the orbit, the server waits in the system for a new customer with probability 1 – \( a \). At the end of vacation, if there are no customers waiting in the orbit, the server waits for the one to arrive. The vacation time of the server \( V \) has distribution function \( V(t) \) and LST \( V^*(x) \). The conditional completion rates for vacation is

\[
\gamma(x)dx = \frac{dV(x)}{1-V(x)}
\]

5. The system may breakdown at random, and the service channel will fail for a short interval of time. The server’s life times are generated by exogenous Poisson Process with rates \( \alpha_1 \) and \( \alpha_2 \) for regular and optional re-service respectively.
6. Once the system breakdown, their repairs do not start immediately and there is a waiting period of the server. We define the waiting time as delay time. The delay time follows a general distribution with density function \( W_1(t) \) for normal service, \( W_2(t) \) for optional repair and LST \( F_{W1}(y), F_{W2}(y) \) respectively. The repair time involves \( F_1(t) \) for normal service, \( F_2(t) \) for optional repair and LST \( F_{W1}(y), F_{W2}(y) \) respectively. The conditional completion rates for delaying repair on normal service, delaying repair on optional repair and LST respectively are

\[
\eta_1(y)dy = \frac{dW_1(y)}{1-W_1(y)}, \quad \eta_2(y)dy = \frac{dW_2(y)}{1-W_2(y)} \quad \text{and} \quad \xi_1(y)dy = \frac{dF_1(y)}{1-F_1(y)}, \quad \xi_2(y)dy = \frac{dF_2(y)}{1-F_2(y)}
\]

III. DEFINITIONS AND NOTATIONS

All stochastic processes involved in the system are assumed to be independent of each other.

Now we introduce some more notations which will be used for the mathematical formulation of this model.

In addition let \( A^0(t), B_1^o(t), B_2^o(t), W_1^0(t), W_2^0(t), F_1^0(t), F_2^0(t), V^0(t) \) be the elapsed retrial time, regular service time, optional repair and LST respectively at time \( t \)

In the steady state, we assume that

\[
A(0) = 0, \quad A(x) = 1, \quad B_1(0) = 0, \quad B_1(x) = 1, \quad B_2(0) = 0, \quad B_2(x) = 1, \quad V(0) = 0, \quad V(x) = 1 \quad \text{are continuous at } x = 0
\]

and \( W_1(0) = 0, \quad W_1(x) = 1, \quad F_1(0) = 0, \quad F_1(x) = 1, \quad W_2(0) = 0, \quad W_2(x) = 1, \quad F_2(0) = 0, \quad F_2(x) = 1 \) are continuous at \( y = 0 \)

The state of system at time \( t \) can be described by the Markov process \( \{C(t), X(t), t \geq 0\} \) where \( C(t) \) denotes the server state 0, 1, 2, 3, 4, 5, 6, 7 according to the server is idle, busy, re-service, delaying repair on normal service, delaying repair on optional repair and LST respectively and \( X(t) \) denotes the repair times involved in the system.
Let \( \{t_n/n \in N \} \) be the sequence of epochs at which either a service period completion occurs or a vacation time ends or a repair period ends. The sequence of random vectors \( Z_n = \{C(t_n+), X(t_n+)\} \) form a Markov chain, which is embedded Markov chain for our retrial queueing system.

Theorem: The embedded Markov chain \( \{Z_n/n \in N \} \) is ergodic if and only if \( \rho < 1 \) where

\[
\rho = E(X)(1-A^*(\lambda)) + \delta E(X)[E(B_1)\alpha_1(E(W_1) + E(F_1)) + r E(B_2)[1+E(W_2) + E(F_2)] + a E(V)]
\]

Ergodicity Condition: We analyze the embedded Markov chain at departure completion epochs

\[ C(t) = \begin{cases} 
0, & \text{if the server is idle at time } t \\
1, & \text{if the server is busy at time } t \\
2, & \text{if the server is on re-service at time } t \\
3, & \text{if the server is delaying repair on normal service at time } t \\
4, & \text{if the server is delaying repair on optional re-service at time } t \\
5, & \text{if the server is repair on normal service at time } t \\
6, & \text{if the server is repair on optional re-service at time } t \\
7, & \text{if the server is on vacation at time } t
\end{cases} \]

The following probabilities are used in sequent sections

\[ P_0(t) = P[C(t) = 0, X(t) = 0] \]

\[ P_n(x,t)dx = P[C(t) = 0, X(t) = n, x \leq A_0(t) < x + dx], \quad n \geq 1 \]

\[ Q_{1,n}(x,t)dx = P[C(t) = 1, X(t) = n, x \leq B_0(t) < x + dx] \text{ for } t \geq 0, x \geq 0, n \geq 0 \]

\[ Q_{2,n}(x,t)dx = P[C(t) = 2, X(t) = n, x \leq B_2(t) < x + dx] \text{ for } t \geq 0, x \geq 0, n \geq 0 \]

\[ D_{1,n}(x,y,t)dy = P[C(t) = 3, X(t) = n, y \leq W_1(t) < y + dy / B_1(t) = x] \]

\[ D_{2,n}(x,y,t)dy = P[C(t) = 4, X(t) = n, y \leq W_2(t) < y + dy / B_2(t) = x] \]

\[ R_{1,n}(x,y,t)dy = P[C(t) = 5, X(t) = n, y \leq F_1(t) < y + dy / B_1(t) = x] \text{ for } t \geq 0, (x, y) \geq 0, n \geq 0 \]

\[ R_{2,n}(x,y,t)dy = P[C(t) = 6, X(t) = n, y \leq F_2(t) < y + dy / B_2(t) = x] \text{ for } t \geq 0, (x, y) \geq 0, n \geq 0 \]

\[ \Omega_n(x,t)dx = P[C(t) = 7, X(t) = n, x \leq V_0(t) < x + dx], \quad n \geq 0 \]

\[ P_0(t) \text{ is the probability that the system is empty at time } t. \]

\[ P_n(x,t) \text{ is the probability that at time } t \text{ there are exactly } n \text{ customers in the orbit with the elapsed service time of the rest customer undergoing retrial is } x. \]

\[ Q_{1,n}(x,t) \text{ is the probability that at time } t \text{ there are exactly } n \text{ customers in the orbit with the elapsed retrial time of the test customer undergoing service is } x. \]
$Q_{2,n}(x,t)$ is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed re-service time of the test customer undergoing service is $x$.

$D_{1,n}(x,y,t)$ is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed service time of the test customer undergoing service is $x$ and the elapsed delaying repair of server is $y$.

$D_{2,n}(x,y,t)$ is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed re-service time of the test customer undergoing service is $x$ and the elapsed delaying repair of server is $y$.

$R_{1,n}(x,y,t)$ is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed service time of the test customer undergoing service is $x$ and the elapsed repair time of server is $y$.

$R_{2,n}(x,y,t)$ is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed service time of the test customer undergoing service is $x$ and the elapsed repair time of server is $y$.

$\Omega_n(x,t)$ is the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed vacation time is $x$.

We assume that the stability condition is fulfilled in the sequence and so that the limiting probabilities

$$P_0 = \lim_{n \to \infty} P_0(t) \text{ for } t \geq 0, x \geq 0, n \geq 1$$

$$Q_{1,n}(x) = \lim_{n \to \infty} Q_{1,n}(x,t) \text{ for } t \geq 0, x \geq 0, n \geq 0, \quad Q_{2,n}(x,t) = \lim_{n \to \infty} Q_{2,n}(x,t) \text{ for } t \geq 0, x \geq 0, n \geq 0$$

$$D_{1,n}(x,y) = \lim_{n \to \infty} D_{1,n}(x,t) \text{ for } t \geq 0, D_{2,n}(x,y) = \lim_{n \to \infty} D_{2,n}(x,t) \text{ for } t \geq 0$$

$$R_{1,n}(x,y) = \lim_{n \to \infty} R_{1,n}(x,t) \text{ for } t \geq 0, R_{2,n}(x,y) = \lim_{n \to \infty} R_{2,n}(x,t) \text{ for } t \geq 0, \quad \Omega_n(x) = \lim_{n \to \infty} \Omega_n(x,t) \text{ for } t \geq 0$$

**IV. STEADY STATE EQUATIONS**

By the method of supplementary variable technique, we obtain the system of equations that govern the dynamics of the system behavior under steady state as

$$\lambda P_0 = (1-a) \left[ \int_0^\infty Q_{1,0}(x)\mu_1(x)dx + \int_0^\infty Q_{2,0}(x)\mu_2(x)dx \right] + \int_0^\infty V_0(x)v(x)dx$$

$$\frac{dP_n(x)}{dx} + (\lambda + a(x))P_n(x) = 0, n \geq 1$$

$$\frac{dQ_{1,0}(x)}{dx} + [\lambda + \alpha_1 + \mu_1(x)]Q_{1,0}(x) = \int_0^\infty \xi_1(y)R_{1,0}(x,y)dy, n = 0$$

$$\frac{dQ_{1,n}(x)}{dx} + [\lambda + \alpha_1 + \mu_1(x)]Q_{1,n}(x) = \lambda \sum_{k=1}^{n} C_k Q_{1,n-k}(x) + \int_0^\infty \xi_1(y)R_{1,n}(x,y)dy, n \geq 1$$

$$\frac{dQ_{2,0}(x)}{dx} + [\lambda + \alpha_2 + \mu_2(x)]Q_{2,0}(x) = \int_0^\infty \xi_2(y)R_{2,0}(x,y)dy, n = 0$$
The steady state boundary conditions are

\[
\frac{dQ_{2,n}(x)}{dx} + \left[\lambda + \alpha_2 + \mu_2(x)\right]Q_{2,n}(x) = \lambda \sum_{k=1}^{n} C_k Q_{2,n-k}(x) + \int_{0}^{\infty} \phi_2(y)R_{2,n}(x,y)dy, \ n \geq 1
\]  \tag{4.6}

\[
\frac{dD_{1,0}(x,y)}{dy} + (\lambda + \eta_1(y))D_{1,0}(x,y) = 0, \ n = 0
\]  \tag{4.7}

\[
\frac{dD_{2,n}(x,y)}{dy} + (\lambda + \eta_1(y))D_{2,n}(x,y) = \lambda \sum_{k=1}^{n} C_k D_{2,n-k}(x,y), \ n \geq 1
\]  \tag{4.8}

\[
\frac{dD_{2,0}(x,y)}{dy} + (\lambda + \eta_2(y))D_{2,0}(x,y) = 0, \ n = 0
\]  \tag{4.9}

\[
\frac{dD_{2,n}(x,y)}{dy} + (\lambda + \eta_2(y))D_{2,n}(x,y) = \lambda \sum_{k=1}^{n} C_k D_{2,n-k}(x,y), \ n \geq 1
\]  \tag{4.10}

\[
\frac{dR_{1,0}(x,y)}{dy} + (\lambda + \xi_1(y))R_{1,0}(x,y) = 0, \ n = 0
\]  \tag{4.11}

\[
\frac{dR_{1,n}(x,y)}{dy} + (\lambda + \xi_1(y))R_{1,n}(x,y) = \lambda \sum_{k=1}^{n} C_k R_{1,n-k}(x,y), \ n \geq 1
\]  \tag{4.12}

\[
\frac{dR_{2,0}(x,y)}{dy} + (\lambda + \xi_2(y))R_{2,0}(x,y) = 0, \ n = 0
\]  \tag{4.13}

\[
\frac{dR_{2,n}(x,y)}{dy} + (\lambda + \xi_2(y))R_{2,n}(x,y) = \lambda \sum_{k=1}^{n} C_k R_{2,n-k}(x,y), \ n \geq 1
\]  \tag{4.14}

\[
\frac{d\Omega_0(x)}{dx} + (\lambda + \gamma(x))\Omega_0(x) = 0, \ n = 0
\]  \tag{4.15}

\[
\frac{d\Omega_n(x)}{dx} + (\lambda + \gamma(x))\Omega_n(x) = \lambda \sum_{k=1}^{n} C_k \Omega_{n-k}(x), \ n \geq 1
\]  \tag{4.16}

The steady state boundary conditions are

\[
P_{n}(0) = \int_{0}^{\infty} V_n(x)\gamma(x)dx + (1-a) \left[ \int_{0}^{\infty} Q_{1,n}(x)\mu_1(x)dx + \int_{0}^{\infty} Q_{2,n}(x)\mu_2(x)dx \right], \ n \geq 1
\]  \tag{4.17}

\[
Q_{1,n}(0) = \int_{0}^{\infty} P_{n+1}(x)a(x)dx + \lambda \sum_{k=1}^{n} C_k \int_{0}^{\infty} P_{n-k}(x)dx + \lambda C_{n+1}P_n, \ n \geq 1
\]  \tag{4.18}

\[
Q_{2,n}(0) = r \int_{0}^{\infty} Q_{1,n}(x)\mu_1(x)dx, \ n \geq 0
\]  \tag{4.19}
\[ D_{1,n}(x,0) = \alpha_1 Q_{1,n}(x), n \geq 0 \]  
(4.20)

\[ D_{2,n}(x,0) = \alpha_2 Q_{2,n}(x), n \geq 0 \]  
(4.21)

\[ R_{1,n}(x,0) = \int_0^\infty D_{1,n}(x,y)\eta_1(y)dy, n \geq 0 \]  
(4.22)

\[ R_{2,n}(x,0) = \int_0^\infty D_{2,n}(x,y)\eta_2(y)dy, n \geq 0 \]  
(4.23)

\[ \Omega_n(0) = r \int_0^\infty Q_{1,0}(x)\mu_1(x)dx + \int_0^\infty Q_{2,0}(x)\mu_2(x)dx, n = 0 \]  
(4.24)

\[ \Omega_n(0) = ar \int_0^\infty Q_{1,n}(x)\mu_1(x)dx + a \int_0^\infty Q_{2,n}(x)\mu_2(x)dx, n \geq 1 \]  
(4.25)

The normalizing condition is

\[ P_o + \sum_{n=1}^{\infty} \int_0^\infty P_n(x)dx + \sum_{n=0}^{\infty} \left[ \int_0^\infty Q_{1,n}(x)dx + \int_0^\infty Q_{2,n}(x)dx + \int_0^\infty \Omega_n(x)dx + \int_0^\infty \int_0^\infty D_{1,n}(x,y)dxdy \right] = 1 \]  
(4.26)

V. QUEUE SIZE DISTRIBUTION

We define the generating functions

\[ P(x, z) = \sum_{n=1}^{\infty} P_n(x)z^n ; \quad P(0, z) = \sum_{n=0}^{\infty} P_n(0)z^n \]  
\[ Q_1(x, z) = \sum_{n=0}^{\infty} Q_{1,n}(x)z^n \quad Q_2(0, z) = \sum_{n=0}^{\infty} Q_{2,n}(0)z^n \quad Q_1(0, z) = \sum_{n=0}^{\infty} Q_{1,n}(0)z^n \]  
\[ Q_2(x, z) = \sum_{n=0}^{\infty} Q_{2,n}(x)z^n \]  
\[ D_1(x, y, z) = \sum_{n=0}^{\infty} D_{1,n}(x,y)z^n \quad D_1(x,0, z) = \sum_{n=0}^{\infty} D_{1,n}(x,0)z^n \]  
\[ D_2(x, y, z) = \sum_{n=0}^{\infty} D_{2,n}(x,y)z^n \quad D_2(x,0, z) = \sum_{n=0}^{\infty} D_{2,n}(x,0)z^n \]  
\[ R_1(x, y, z) = \sum_{n=0}^{\infty} R_{1,n}(x,y)z^n \quad R_1(x,0, z) = \sum_{n=0}^{\infty} R_{1,n}(x,0)z^n \]  
\[ R_2(x, y, z) = \sum_{n=0}^{\infty} R_{2,n}(x,y)z^n \quad R_2(x,0, z) = \sum_{n=0}^{\infty} R_{2,n}(x,0)z^n \]  
\[ \Omega(x, z) = \sum_{n=0}^{\infty} \Omega_n(x)z^n ; \quad \Omega(0, z) = \sum_{n=0}^{\infty} \Omega_n(0)z^n \]  
\[ \Omega(x, z) = \sum_{n=0}^{\infty} \Omega_n(x)z^n \]  
\[ \Omega(0, z) = \sum_{n=0}^{\infty} \Omega_n(0)z^n \]  

Now multiply the steady state equation and steady state boundary conditions (4.2) to (4.25) by \( z^n \) and summing over \( n \).
\[
\frac{dP(x, z)}{dx} + (\lambda + a(x))P(x, z) = 0
\]  
(5.1)

\[
\frac{dQ_1(x, z)}{dx} + [\lambda - \lambda X(z) + \alpha_1 + \mu_1(x)]Q_1(x, z) = \int_0^\infty \xi_1(y)R_1(x, y, z)dy
\]  
(5.2)

\[
\frac{dQ_2(x, z)}{dx} + [\lambda - \lambda X(z) + \alpha_2 + \mu_2(x)]Q_2(x, z) = \int_0^\infty \xi_2(y)R_2(x, y, z)dy
\]  
(5.3)

\[
\frac{dD_1(x, y, z)}{dy} + [\lambda - \lambda X(z) + \eta_1(y)]D_1(x, y, z) = 0
\]  
(5.4)

\[
\frac{dD_2(x, y, z)}{dy} + [\lambda - \lambda X(z) + \eta_2(y)]D_2(x, y, z) = 0
\]  
(5.5)

\[
\frac{dR_1(x, y, z)}{dy} + [\lambda - \lambda X(z) + \xi_1(y)]R_1(x, y, z) = 0
\]  
(5.6)

\[
\frac{dR_2(x, y, z)}{dy} + [\lambda - \lambda X(z) + \xi_2(y)]R_2(x, y, z) = 0
\]  
(5.7)

\[
\frac{dQ(x, z)}{dx} + [\lambda - \lambda X(z)]Q(x, z) = 0
\]  
(5.8)

The steady state boundary conditions at \( x = 0 \) and \( y = 0 \) are

\[
P(0, z) = \int_0^\infty Q(x, z)\gamma(x)dx + (1 - a)\left[ \int_0^\infty Q_1(x, z)\mu_1(x)dx + \int_0^\infty Q_2(x, z)\mu_2(x)dx \right] - \lambda P_0
\]  
(5.9)

\[
Q_1(0, z) = \frac{1}{z} \int_0^\infty P(x, z)a(x)dx + \frac{\lambda X(z)}{z} \int_0^\infty P(x, z)dx + \lambda P_0
\]  
(5.10)

\[
Q_2(0, z) = r \int_0^\infty Q_1(x, z)\mu_1(x)dx
\]  
(5.11)

\[
D_1(x, 0, z) = \alpha_1 Q_1(x, z), n \geq 0
\]  
(5.12)

\[
D_2(x, 0, z) = \alpha_2 Q_2(x, z), n \geq 0
\]  
(5.13)

\[
R_1(x, 0, z) = \int_0^\infty D_1(x, y, z)\eta_1(y)dy
\]  
(5.14)

\[
R_2(x, 0, z) = \int_0^\infty D_2(x, y, z)\eta_2(y)dy
\]  
(5.15)
\[ \Omega(0, z) = a^r \int_0^\infty Q_1(x, 0)\mu_1(x)dx + a^\gamma \int_0^\infty Q_2(x, 0)\mu_2(x)dx, n \geq 0 \]  

(5.16)

Solving the above PDE (5.1) to (5.16), it follows that

\[ P(x, z) = P(0, z)[1 - A(x)]e^{-\lambda x} \]  

(5.17)

\[ Q_1(x, z) = Q_1(0, z)[1 - B_1(x)]e^{-A_1(z)x} \]  

(5.18)

\[ Q_2(x, z) = Q_2(0, z)[1 - B_2(x)]e^{-A_2(z)x} \]  

(5.19)

\[ D_1(x, y, z) = D_1(x, 0, z)[1 - W_1(y)]e^{-h(z)y} \]  

(5.20)

\[ D_2(x, y, z) = D_2(x, 0, z)[1 - W_2(y)]e^{-h(z)y} \]  

(5.21)

\[ R_1(x, y, z) = R_1(x, 0, z)[1 - F_1(y)]e^{-h(z)y} \]  

(5.22)

\[ R_2(x, y, z) = R_2(x, 0, z)[1 - F_2(y)]e^{-h(z)y} \]  

(5.23)

\[ \Omega(x, z) = \Omega(0, z)[1 - V(x)]e^{-h(z)x} \]  

(5.24)

Where \( A_1(z) = h(z) + \alpha_1[1 - W_1^*(h(z))]F_1^*(h(z)) \), \( A_2(z) = h(z) + \alpha_2[1 - W_2^*(h(z))]F_2^*(h(z)) \) and \( h(z) = \lambda(1 - X(z)) \)

Inserting (5.17) in (5.10), we obtain

\[ Q_1(0, z) = \frac{P(0, z)}{z} [X(z) + (1 - X(z))A^*(\lambda)] + \frac{\lambda X(z)}{z} P_0 \]  

(5.25)

Inserting (5.18) in (5.11), we obtain

\[ Q_2(0, z) = rQ_1(0, z)B_1^*(A_1(z)) \]  

(5.26)

Inserting (5.18) in (5.12), we obtain

\[ D_1(x, 0, z) = \alpha_1Q_1(0, z)[1 - B_1(x)]e^{-A_1(z)x} \]  

(5.27)

Inserting (5.19) in (5.13), we obtain

\[ D_2(x, 0, z) = \alpha_2Q_2(0, z)[1 - B_2(z)]e^{-A_2(z)x} \]  

(5.28)

Inserting (5.20) in (5.14), we obtain

\[ R_1(x, 0, z) = D_1(x, 0, z)W_1^*(h(z)) \]  

(5.29)

Inserting (5.21) in (5.15), we obtain

\[ R_2(x, 0, z) = D_2(x, 0, z)W_2^*(h(z)) \]  

(5.30)
Inserting (5.24) in (5.16), we get

$$\Omega(0, z) = arQ_1(0, z)B_1^*(A_1(z)) + aQ_2(0, z)B_2^*(A_2(z))$$

(5.31)

Inserting (5.18), (5.19), (5.24) in (5.9), we get

$$P(0, z) = \frac{Nr(z)}{Dr(z)}$$

(5.32)

$$Nr(z) = \lambda P_0 \left\{ X(z)B_1^*(A_1(z))\left[ arV^*(\lambda_1(z)) + (1 - a)rB_2^*(A_2(z)) + (arV^*(\lambda_1(z)) + (1 - a)r) \right] - z \right\}$$

$$Dr(z) = \left\{ 1 - X(z) + (1 - X(z))A^*(\lambda) \right\} \left\{ arV^*(\lambda_1(z)) + (1 - a)rB_2^*(A_2(z)) + (arV^*(\lambda_1(z)) + (1 - a)r) \right\}$$

Inserting (5.32) in (5.25), we obtain

$$Q_1(0, z) = \lambda P_0 \left\{ (X(z) - 1)A^*(\lambda) \right\} / Dr(z)$$

(5.33)

Inserting (5.33) in (5.26), we obtain

$$Q_2(0, z) = \lambda P_0 \left\{ r(X(z) - 1)A^*(\lambda)B_1^*(A_1(z)) \right\} / Dr(z)$$

(5.34)

Inserting (5.33) in (5.27), we obtain

$$D_1(x, 0, z) = \lambda P_0 \alpha_1 \left\{ (X(z) - 1)A^*(\lambda)(1 - B_1(x))e^{-A_1(z)x} \right\} / Dr(z)$$

(5.35)

Inserting (5.34) in (5.28), we obtain

$$D_2(x, 0, z) = r\lambda P_0 \alpha_2 \left\{ (X(z) - 1)A^*(\lambda)B_1^*(A_1(z))(1 - B_2(x))e^{-A_1(z)x} \right\} / Dr(z)$$

(5.36)

Inserting (5.35) in (5.29), we obtain

$$R_1(x, 0, z) = \lambda P_0 \alpha_1 \left\{ (X(z) - 1)A^*(\lambda)(1 - B_1(x))e^{-A_1(z)x}W_1^*(h(z)) \right\} / Dr(z)$$

(5.37)

Inserting (5.36) in (5.30), we obtain

$$R_2(x, 0, z) = \lambda P_0 \alpha_2 \left\{ (X(z) - 1)A^*(\lambda)B_1^*(A_1(z))(1 - B_2(x))e^{-A_1(z)x}W_2^*(h(z)) \right\} / Dr(z)$$

(5.38)

Inserting (5.33), (5.34) in (5.31), we obtain

$$\Omega(0, z) = \lambda P_0 \left\{ X(z) - 1 \right\} A^*(\lambda) \left\{ arB_1^*(A_1(z)) + arB_1^*(A_1(z))B_2^*(A_2(z)) \right\} / Dr(z)$$

(5.39)

Using (5.32) - (5.39) in (5.17) - (5.24) then we get the limiting probability generating functions

$$P(x, z), Q_1(x, z), Q_2(x, z), D_1(x, y, z), D_2(x, y, z), R_1(x, y, z), R_2(x, y, z), \Omega(x, z)$$
\[ P(x, z) = \lambda P_{z} \left[ \frac{N_r(z)}{D_r(z)} \right] (1 - A(x))e^{-\lambda x} \]  
\[ N_r(z) = X(z)B_r^*(A_r(z))[arV^*(h(z)) + (1-a)r]B_r^*(A_r(z)) + (arV^*(h(z)) + (1-a)r] - z \]  
\[ D_r(z) = \left[ z - [X(z) + (1 - X(z))A^*(\lambda)]B_r^*(A_r(z))[arV^*(h(z)) + (1-a)r]B_r^*(A_r(z)) + (arV^*(h(z)) + (1-a)r)] \right] \]  
\[ Q_i(x, z) = \lambda P_0 \left[ (X(z) - 1)A^*(\lambda) \right] (1 - D_i(x))e^{-\lambda(z)x} / Dr(z) \]  
\[ Q_2(x, z) = r\lambda P_0 [(X(z) - 1)A^*(\lambda)]B_r^*(A_r(z))(1 - B_2(x))e^{-\lambda(z)x} / Dr(z) \]  
\[ D_1(x, y, z) = \lambda P_0 \alpha_1 \left\{ (X(z) - 1)A^*(\lambda)(1 - W_1(y))e^{-\lambda(z)x} (1 - B_1(x))e^{-\lambda(z)x} \right\} / Dr(z) \]  
\[ D_2(x, y, z) = \lambda P_0 \alpha_2 r \left\{ (X(z) - 1)A^*(\lambda)(1 - W_2(y))e^{-\lambda(z)x} (1 - B_2(x))e^{-\lambda(z)x} \right\} / Dr(z) \]  
\[ R_1(x, y, z) = \lambda P_0 \alpha_1 \left\{ (X(z) - 1)A^*(\lambda)(1 - F_1(y))e^{-\lambda(z)x} W_1^*(h(z))(1 - B_1(x))e^{-\lambda(z)x} \right\} / Dr(z) \]  
\[ R_2(x, y, z) = \lambda P_0 \alpha_2 r \left\{ (X(z) - 1)A^*(\lambda)(1 - F_2(y))e^{-\lambda(z)x} W_2^*(h(z))(1 - B_2(x))e^{-\lambda(z)x} B_r^*(A_r(z)) \right\} / Dr(z) \]  
\[ \Omega(x, z) = \lambda P_0 \left\{ (X(z) - 1)A^*(\lambda) \right\} (ar + \alpha B_r^*(A_r(z))B_r^*(A_r(z))(1 - V(x))e^{-\lambda(z)x}) / Dr(z) \]  

The marginal orbit size distributions due to system state of the server. The limiting probability generating functions, \( P(x, z), Q_1(x, z), Q_2(x, z), D_1(x, y, z), D_2(x, y, z), R_1(x, y, z), R_2(x, y, z), \Omega(x, z) \)

We define the partial probability generating functions as

\[ P(z) = \int_0^\infty P(x, z)dx, Q_1(z) = \int_0^\infty Q_1(x, z)dx, Q_2(z) = \int_0^\infty Q_2(x, z)dx, D_1(z) = \int_0^\infty D_1(x, z)dx, D_2(z) = \int_0^\infty D_2(x, z)dx, R_1(z) = \int_0^\infty R_1(x, z)dx, R_2(z) = \int_0^\infty R_2(x, z)dx, \Omega(z) = \int_0^\infty \Omega(x, z)dx \]

Note that \( P(z), Q_1(z), Q_2(z), D_1(z), D_2(z), R_1(z), R_2(z), \Omega(z) \) is the probability function of orbit size when the server is idle, busy, re-service, repair on normal service, repair on re-service, delaying repair on normal service, delaying repair on re-service, on vacation respectively.

\[ P(z) = \frac{N_r(z)}{D_r(z)} \]  
\[ N_r(z) = P_r(1 - A^*(\lambda))[X(z)B_r^*(A_r(z))[arV^*(h(z)) + (1-a)r]B_r^*(A_r(z)) + (arV^*(h(z)) + (1-a)r] - z \]  
\[ D_r(z) = \left[ z - [X(z) + (1 - X(z))A^*(\lambda)]B_r^*(A_r(z))[arV^*(h(z)) + (1-a)r]B_r^*(A_r(z)) + (arV^*(h(z)) + (1-a)r)] \right] \]  
\[ Q_1(z) = P_0 \left( \frac{B_r^*(A_r(z)) - 1}{A_r(z)} \right) h(z)A^*(\lambda) / Dr(z) \]  

(5.48)  

(5.49)
We derive the system performance of our model. We define the probability generating functions of the number of customers in the orbit is applying L-Hospital’s rule whenever necessary and we get

\[
P_0 = E(X)\left[A^*(\lambda) - \lambda E(B_1)[1 + \alpha_1(E(W_1) + E(F_1))] + r E(B_2)[1 + \alpha_2(E(W_2) + E(F_2)) + aE(V)]\right] / A^*(\lambda)
\]

(5.56)

We define the probability generating functions of the number of customer in the system is

\[
K(z) = P_0 + P(z) + \Omega(z) + z\left[Q_1(z) + Q_2(z) + D_1(z) + D_2(z) + R_1(z) + R_2(z)\right]
\]

(5.57)

We define the probability generating functions of the number of customers in the orbit is

\[
H(z) = P_0 + P(z) + \Omega(z) + Q_1(z) + Q_2(z) + D_1(z) + D_2(z) + R_1(z) + R_2(z)
\]

(5.58)

VI. PERFORMANCE MEASURES

We derive the system performance of our model.

The mean number of customers in the system \(L\), under steady state condition is obtained by differentiating (5.57) with respect to \(z\) and evaluating at \(z = 1\)

\[
L_s = \lim_{z \to 1} K'(z)
\]
$$L_q = \frac{2A^*(\lambda) - E(X)(rE(B_1)(1 + \alpha_1(E(W_1) + E(F_1))) + aE(V))}{2\lambda E(X)(rE(B_1)(1 + \alpha_1(E(W_1) + E(F_1))) + rE(B_2)(1 + \alpha_2(E(W_2) + E(F_2))) + + aE(V))} + A^*(\lambda)$$

The mean number of customers in the system $L_q$ under steady state condition is obtained by differentiating (5.58) with respect to $z$ and evaluating at $z = 1$

$$L_q = \lim_{z \to 1} H'(z)$$

$$L_q = \frac{1 - A^*(\lambda)\lambda E(X)[E(B_1)(1 + \alpha_1(E(W_1) + E(F_1))) + rE(B_2)(1 + \alpha_2(E(W_2) + E(F_2)) + aE(V))] - \lambda E(X)^2\begin{bmatrix} E(B_1^2)(1 + \alpha_1(E(W_1) + E(F_1))^2 + rE(B_2^2)(1 + \alpha_2(E(W_2) + E(F_2))^2 + aE(V)^2) \\
2\alpha_1 E(B_1)(1 + \alpha_1(E(W_1) + E(F_1)))E(B_2)(1 + \alpha_2(E(W_2) + E(F_2))) + 2\alpha_2 E(B_2)(1 + \alpha_2(E(W_2) + E(F_2)))E(V) + 2\alpha_2 E(B_2)(1 + \alpha_2(E(W_2) + E(F_2)))E(V) - 1 \end{bmatrix}}{2\lambda E(X)}$$

The average time a customer spends in the system $W_1$ and in the orbit $W_2$ under steady state condition due to Little’s formula, we obtain $W_1 = L_q / \lambda E(X)$ and $W_2 = L_q / \lambda E(X)$

We obtain other performance measures

Let $U$ be the steady state probability that the server is busy (server utilization), $I$ be the steady state probability that the server is idle during the retrial time or on vacation

$$U = Q_1(z) + Q_2(z) = \frac{P \cdot A^*(\lambda)\lambda E(X)[E(B_1)(1 + \alpha_1(E(W_1) + E(F_1))) + rE(B_2)(1 + \alpha_2(E(W_2) + E(F_2)))]}{A^*(\lambda) - \lambda E(X)[E(B_1)(1 + \alpha_1(E(W_1) + E(F_1))) + rE(B_2)(1 + \alpha_2(E(W_2) + E(F_2)))] + aE(V)}$$

$$I = P_1 + P(1) + V(1) = \frac{P \cdot A^*(\lambda)[\lambda E(X)[E(B_1)(1 + \alpha_1(E(W_1) + E(F_1))) + rE(B_2)(1 + \alpha_2(E(W_2) + E(F_2)))] - 1}{A^*(\lambda) - \lambda E(X)[E(B_1)(1 + \alpha_1(E(W_1) + E(F_1))) + rE(B_2)(1 + \alpha_2(E(W_2) + E(F_2)))] + aE(V)}$$

VI.1 SPECIAL CASES

In this section, special cases of our model can be deduced.

Case(i): Here we assume that there is no re-service, then $r = 0$. Our queue size distribution reduces to

$$K(z) = \frac{P \cdot B_1^*(A_1(z))(z - 1)}{z - [(X(z) + (1 - X(z)))A^*(\lambda)B_1^*(A_1(z))aV^*(b(z)) + (1 - a)]}$$

Where $P = \frac{E(X)[A^*(\lambda) - \lambda E(B_1)(1 + \alpha_1(E(W_1) + E(F_1)))] + aE(V)]}{A^*(\lambda)}$
Case(ii): No vacation, No breakdown, No retrial this model can be reduced to the following form

Let $a = 1, \alpha_1 = \alpha_2 = 0, A^*(\lambda) \to 1$

$$K(z) = \frac{P \cdot B^*_1(h(z))(z-1) - (r + rB^*_2(h(z)))}{z - (B^*_1(h(z))(r + rB^*_2(h(z)))W^*(h(z)))}$$

where $P \cdot E(X)(1 - \lambda[E(B_1) + rE(B_2) + E(V)])$

Case(iii): Single Poisson arrival, No vacation, No breakdown and No optional re-service then our model can be reduced to $M/G/1$ retrial queue. The following form and results agree with Gomez-Corral(1999)

$$K(z) = \frac{P \cdot A^*(\lambda) B^*_1(h(z))(z-1)}{z - \{X(z) + (1 - X(z)) A^*(\lambda) B^*_1(h(z))\}}$$

where $P \cdot A^*(\lambda) - \lambda[E(B_1)]$

VII. CONCLUSION

In this paper, we have discussed a single server queue with customers batches in a system of variable size, but are served one by one. Further we assume that the service times, vacation times, delay times, repair times each have a general distribution. We derived the probability generating functions of number of customers in the orbit for different states. Various performance measures and special cases have analyzed. The results of this work finds applications in mailing system, software designs of one by one. Further we assume that the service times, vacation times, delay times, repair times each have a general distribution. This work can also be extended further by incorporating, the concepts of impatient customers, Balking / Reneging, Orbit search, Starting failure, Working Vacation policies.

REFERENCES