Commutativity of prime rings with symmetric biderivations on ideals

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Abstract: A biadditive mapping $B_{1}(\cdot,\cdot): R \times R \rightarrow R$ is called a symmetric biderivation if $B_{1}(x,y) = B_{1}(y,x)$ for all $x,y \in R$. In this paper, we initiated the commutativity of a prime ring $R$ with a nonzero left ideal $I$ of $R$, which satisfies certain conditions, namely $B_{1}(u,m)ou \in Z(R), \{B_{1}(u,m),B_{1}(v,m)\} \in Z(R), B_{1}(u,v,m) + B_{1}(u,m,v) \in Z(R), B_{1}(u,v,m) \in Z(R), B_{1}(u,v,m) \in Z(R)$.

1. INTRODUCTION

In recent years, several authors have investigated the concept of commutativity of prime rings by considering derivations (see [2],[3],[8],[11] and [12]), as well as commutativity of prime rings by considering generalized derivations (see [4],[9] and [10]). The concept of derivations with prime rings was initiated by E.C. Posner[1] in the year 1957. H.E. Bell and S. Martindale III[2] exposed that a semiprime ring $R$ must have a nontrivial central ideal if it admits an appropriate endomorphism or derivation which centralizing on some nontrivial one sided ideal. I.N. Herstein[5] determined the structure of a prime ring $R$ which has a derivation $d \neq 0$ such that the values of $d$ commute i.e for which $d(x)d(y) = d(y)d(x)$, for any $x,y \in R$. Shaker Ali and H Shuliang[14] produced the commutativity results for rings and presented that if $R$ is a 2-torsion free semiprime ring and I a nonzero ideal of $R$, then a derivation $d$ of $R$ is commuting on $I$ if one of the following conditions holds i. $d(x)d(y) = xoy$ , ii. $d(x)d(y) = -xoy$, iii. $d(x)d(y) = 0$, iv. $[d(x),d(y)] = -[x,y]$. That is, $d$ commute.

2. PRELIMINARIES

In each part of this article all rings assumed to be associative and possesses an identity with center of a ring $R$ is $Z(R)$. As a wellknown fact every $x,y \in R$, the commutator ($xy - yx$) be denoted by $[x,y]$ and the anticommutator ($xy + yx$) be denoted by $xoy$. Recall that $R$ is a prime ring if $xRx = 0$ implies $x = 0$ or $y = 0$ and is a semiprime if $xRx = 0$ implies $x = 0$. An additive map $d: R \rightarrow R$ is called derivation if $d(xd(y)) = d(x)y + xd(y)$ for all $x, y \in R$. A biadditive mapping $B(\cdot, \cdot): R \times R \times R$ is called symmetric biderivation if $B_{1}(x,y) = B_{1}(y,x)$, for any $x,y \in R$. A biadditive mapping $B_{1}(\cdot, \cdot): R \times R \times R$ is called a symmetric biderivation if $B_{1}(x,y,z) = B_{1}(y,x,z)$ for all $x,y,z \in R$. Obviously, in this case also if $B_{1}(x,yz) = B_{1}(y,xz) + B_{1}(x,y,z)$, for all $x,y,z \in R$. We use without mention, the following are basic commutator and anticommutator identities

$[x,y] = x[y,z] + [x,z]y$, $[x,y]z = [x,y]z + y[x,z]$, $xo(yz)z - y[x,z] = y(xoz) + [x,y]z$. 

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\[(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z].\]

The following results are very significant to prove main theorems.

**Remark:** For a prime ring \( R \), a nonzero element \( a \in Z(R) \), if \( ab \in Z(R) \), then \( b \in Z(R) \).

Let us begin our discussion with the following lemmas

**Lemma 2.1:** For a prime ring \( R \), if \( B_1: R \times R \to R \) is a symmetric biderivation on \( R \), then for any \( w \neq 0 \in Z(R) \), \( B_1(w, m) \in Z(R) \).

**Proof:** For any \( w \neq 0 \in Z(R) \) which indicates \([w, r] = 0\), for any \( r \in R \) and hence we write \( B_1([w, r], m) = 0 \), for any \( m \in I \), then using commutator identity to find that \( B_1([w, r], m) = B_1(w, r) - B_1(r, m)w - rB_1(w, m) = 0 \). Since \( w \in Z(R) \), \( B_1(r, m) = B_1(r, m)w \). Therefore \( B_1(w, m)r - rB_1(w, m) = 0 \), which implies \( B_1(w, m), r = 0 \). From this it is clear that \( B_1(w, m) \in Z(R) \).

**Lemma 2.2:** Let \( I \) be a nonzero left ideal of a prime ring \( R \) and if \( B_1 \) is a symmetric biderivation of \( R \) which centralizing on \( I \) then \( B_1 \) is commuting on \( I \).

**Proof:** For any \( u \in I \), \( [x, B_1(u, m)] \in Z(R) \), also \([u^2, B_1(u^2, m)] \in Z(R) \). Simplifying, we get \([u^2, uB_1(u, m)] \in Z(R) \). We shall rewritten it as \([u^2, 2uB_1(u, m)] \in Z(R) \). Then it reduces to \(4[u^2, B_1(u, m)]\{B_1(u, m)\} = 0 \) and 8\{B_1(u, m)\} = 0. Hence \{B_1(u, m)\} = 0 or \{B_1(u, m)\} = 0. Using the property (ii) in [2], the center contains no nonzero nilpotent elements. Therefore \(2[B_1(u, m)] = 0 \) and it follows that \([u^2, B_1(u, m)] = 0 \).

By linearizing both Eq. (1) and our original hypothesis, we see that \([u, B_1(u, m)] + [v, B_1(u, m)] \in Z(R) \) and \(2[u, B_1(v, m)] + [v, B_1(u, m)] = 0 \) and combining these results with Eq. (1), we easily see that \([uv + vu, B_1(u, m)] + [u^2, B_1(v, m)] = 0 \).

Changing \( u + v \) with \( u \), we get

\[ [uv + vu, B_1(u, m)] + [u^2, B_1(v, m)] = 0 \]

changing \( u + v \) with \( v \), we get \([u^2, B_1(u, m)] + [v^2, B_1(u, m)] = 0 \). Using the same tricks as used here to prove the below lemma.

**Lemma 2.3:** For a prime ring \( R \) and a nonzero left ideal \( I \) of \( R \), if \( R \) admits a nonzero symmetric biderivation \( B_1 \) such that \([u, B_1(u, m)] \in Z(R) \) for all \( u, m \in I \) then \( R \) is commutative.

**Lemma 2.4:** For a prime ring \( R \), contains a commutative nonzero right ideal, then \( R \) is commutative.

**Lemma 2.5:** (Lemma 2.5): For a prime ring \( R \) and a nonzero left ideal \( I \) of \( R \) which satisfies one of the following conditions

I. \([u, v] \in Z(R) \) or

II. \( uov \in Z(R) \), for any \( u, v \in I \). Then \( R \) is commutative.

3. **MAINRESULTS**

**Theorem 3.1:** For a prime ring \( R \) and a nonzero left ideal \( I \) of \( R \), suppose that \( R \) admits a symmetric biderivation \( B_1 \) such that \( B_1(Z(R), m) \neq 0 \) further, if it satisfies the condition \( B_1(u, m)ou \in Z(R) \), for any \( u, m \in I \), then \( R \) is commutative.

**Proof:** Consider \( B_1(u, m)ou \in Z(R) \), for any \( u, m \in I \).

Substituting \( v \) with \( v \), then \( B_1(u + v, m)ou \in Z(R) \), we obtain

\[ B_1(u, m)ov + B_1(v, m)ou \in Z(R). \] (4)

Since \( B_1(Z(R), m) \neq 0 \), there exist \( w \in Z(R) \) such that \( B_1(w, m) \neq 0 \). Replacing \( v \) by \( wv \) in (4) and using (4), then \( B_1(u, m)owv + B_1(wv, m)ou \in Z(R) \), we attain \( B_1(w, m)(vou) - B_1(w, m), u \in Z(R) \). Therefore using lemma 2.1, \( B_1(w, m) \neq 0 \in Z(R) \) hence we get \( B_1(w, m)(vou) \in Z(R) \). Since \( R \) is prime ring and using remark we find that \( vou \in Z(R) \) and hence by lemma 2.5(II), \( R \) is commutative.

**Theorem 3.2:** For a prime ring \( R \) and a nonzero left ideal \( I \) of \( R \), suppose that if \( R \) admits a symmetric biderivation \( B_1 \) such that \( B_1(Z(R), m) \neq 0 \) further if \( R \) satisfies any one of the following conditions:

i. \( B_1(u, m)b - B_1(v, m) \in Z(R) \)

ii. \( B_1(u, m)b - B_1(u, m) \in Z(R) \), for any \( u, v, m \in I \). Then \( R \) is commutative.

**Proof:** (i) Consider \( B_1(u, m) - B_1(v, m) \)

(5)

Since \( B_1(Z(R), m) \neq 0 \), there exist \( w \in Z(R) \) such that \( B_1(w, m) \neq 0 \). Substituting \( v \) by \( wv \) in (5) and by Eq. (5), we get the expression
\[ B_1(u, m, w)B_2(v, m) + w[B_1(u, m), B_2(v, m)] + B_2(u, m)[B_1(v, m), w] + B_1(v, m)[B_2(u, m), w] = [u, w]B_1(v, m) - B_2(u, m, w)v + wB_1([u, v], m) - B_1(w, m)[u, v] \in Z(R) \text{, since } w \text{ is in center which commute with every element, then which reduces to}\]

\[ w[B_2(u, m), B_1(v, m)]v + B_1(v, m)[B_2(u, m), w] - B_1(w, m)[u, v] \in Z(R) \]

Using lemma 2.1, \( B_1(v, m) \neq 0 \in Z(R) \), so we get \( B_2(u, m)([B_2(u, m), v] - [u, v]) \in Z(R) \).

Since \( B_1(w, m) \in Z(R) \), using remark, we find \([B_1(u, m), v] - [u, v] \in Z(R)\).

Replace \( v \) by \( B_1(w, m)u \) in the above expression and use above expression to get

\[ [B_1(u, m), B_1(v, m)u] - [u, B_1(w, m)u] \in Z(R) \]

\[ B_1(u, m), B_1(v, m)u + B_1(w, m)[B_1(u, m), u] - [u, B_1(w, m)u] \in Z(R) \]

On simplifying, we have \( B_1(w, m)[B_1(u, m), u] \in Z(R) \) using primeness of \( R \), remark we get \([B_1(u, m), u] \in Z(R)\), then using Lemma 2.3, we get \( R \) is commutative.

(ii) consider \( B_1(u, m)oB_1(v, m) - B_1(uov, m) \in Z(R) \)

Since \( B_1(Z(R), m) \neq 0 \), there exist \( w \in Z(R) \) such that \( B_1(w, m) \neq 0 \)

Replacing \( v \) by \( wv \) in the above Eq. (6) and using the Eq. (6), we get

\[ B_1(u, m)oB_1(wv, m) - B_1(uowv, m) \in Z(R) \text{then we see the relation obtained}\]

\[ B_1(u, m)oB_1(wv, m) + [B_1(u, m), B_1(wv, m)] + [B_1(u, m), w]B_1(v, m) - B_1(w, m)[u, w]B_1(v, m) - B_1([u, w], v) \in Z(R) \]

Which gives \( B_1(w, m)(B_1(u, m)ov) + [B_1(u, m), B_1(wv, m)]v - B_1(w, m)(uov) \in Z(R) \).

Using lemma 2.1, \( B_1(w, m) \in Z(R) \) then above expression becomes \( B_1(w, m)((B_1(u, m)ov) - (uov)) \in Z(R) \), from the remark, we have seen that \( B_1(u, m)ov - uov \in Z(R) \), then replacing \( u \) by \( wu \), to get

\[ B_1(w, m)(uov) - B_1([u, w], v) + wB_1(u, m)ov - [w, v]B_1(u, m) - wuov + [w, vu] \in Z(R) \]

Using the similar arguments as used above, it yields to \( B_1(w, m)(uov) \in Z(R) \) and thus we conclude \( uov \in Z(R) \).

So using lemma 2.5(II), it is clear that \( R \) is commutative.

**Theorem 3.3:** For a prime ring \( R \) and a nonzero left ideal \( I \) of \( R \), suppose that \( R \) admits a symmetric biderivation \( B_1 \) such that \( B_1(Z(R), m) \neq 0 \), further if \( R \) satisfies the condition \( B_1([u, v], m) + [B_1(u, m), v] - [B_1(u, m), B_1(v, m)] \in Z(R) \), for any \( u, v, m \in I \), then \( R \) is commutative.

**Proof:** For any \( u, v, m \in I \), we have

\[ B_1([u, v], m) + [B_1(u, m), v] - [B_1(u, m), B_1(v, m)] \in Z(R) \].

Since \( B_1(Z(R), m) \neq 0 \), there exist \( w \in Z(R) \) such that \( B_1(w, m) \neq 0 \)

Replacing \( v \) by \( wv \) in the above Eq. (7) and using the Eq. (7), we get

\[ B_1([u, v], m) + [B_1(u, m), v] - [B_1(u, m), B_1(v, m)] \in Z(R) \]

Since \( B_1(Z(R), m) \neq 0 \), there exist \( w \in Z(R) \) such that \( B_1(w, m) \neq 0 \)

Replacing \( v \) by \( wv \) in the above Eq. (8) and using the Eq. (8), we get

\[ B_1([u, v], m) + [B_1(u, m), v] - [B_1(u, m), B_1(v, m)] \in Z(R) \]

Since \( B_1(Z(R), m) \neq 0 \), there exist \( w \in Z(R) \) such that \( B_1(w, m) \neq 0 \)

Replacing \( v \) by \( wv \) in the above Eq. (8) and using the Eq. (8), we get

\[ B_1([u, v], m) + [B_1(u, m), v] - [B_1(u, m), B_1(v, m)] \in Z(R) \]

Which gives \( B_1(w, m)(uov) \in Z(R) \) and since \( B_1(Z(m), w) \in Z(R) \) by lemma 2.1 and use of remark, we get \( uov \in Z(R) \), then we use the lemma 2.5(II) to get the required result.

(ii) proof of this condition follows using the similar arguments as done in (i).

**Theorem 3.5:** For a prime ring \( R \) and a nonzero left ideal \( I \) of \( R \), suppose that \( R \) admits a symmetric biderivation \( B_1 \) such that \( B_1(Z(R), m) \neq 0 \), further if \( R \) satisfies one of the following condition

i. \( B_1(uov, m) - [u, v] \in Z(R) \)

ii. \( B_1(uov, m) + [u, v] \in Z(R) \), for any \( u, v, m \in I \), then \( R \) is commutative.

**Proof:** (i) we have \( B_1(uov, m) - [u, v] \in Z(R) \)

If \( B_1 = 0 \) then \( uov \in Z(R) \) and hence by lemma 2.5(III), we get the required result.

Therefore we assume that \( B_1 \neq 0 \), for every \( u, v, m \in I \)

Consider \( B_1(uov, m) - [u, v] \in Z(R) \)
Since $B_1(Z(R), m) \neq 0$, there exist $w \in Z(R)$ such that $B_1(w, m) \neq 0$.

Substituting $w$ by $uw$ in the above Eq.(9) and using the Eq.(9), we get

$$B_1(w, m)\{B_1(u, m), v\} + [B_1(u, m), B_1(w, m)]v + [B_1(u, m), w]B_1(v, m) + w[B_1(u, m), B_1(v, m)] - \quad w(uov) + [u, w]v \in Z(R),$$

using the relation $w$ is the element in center of $R$.

Since $B_1(w, m) \in Z(R)$ by lemma 2.1, the above expression becomes $B_1(w, m)[B_1(u, m), v] \in Z(R)$ and using remark, we get $[B_1(u, m), v] \in Z(R)$, in particular we have $[B_1(u, m), u] \in Z(R)$ and we conclude that from lemma 2.3, $R$ is commutative.

(ii) proof of (ii) follows by proceeding the same arguments as above (i).

**Theorem 3.6:** For a prime ring $R$ and a nonzero left ideal $I$ of $R$, suppose that $R$ admits a symmetric biderivations $B_1$ and $D_1$ such that $B_1(Z(R), m) \neq 0$ and $D_1(Z(R), m) \neq 0$, if $R$ satisfies one of the following conditions

i. $[B_1(u, m), D_1(v, m)] - [u, v] \in Z(R)$

ii. $[B_1(u, m), D_1(v, m)] + [u, v] \in Z(R)$, for every $u, v, m \in I$, then $R$ is commutative.

**Proof:** (i) first assume that the condition $[B_1(u, m), D_1(v, m)] - [u, v] \in Z(R)$. If $B_1 = 0$ (or $D_1 = 0$), then $[u, v] \in Z(R)$. Hence by lemma 2.5(I), $R$ is commutative.

Next we assume that, for $B_1$ and $D_1$ are nonzero symmetric biderivations such that $[B_1(u, m), D_1(v, m)] - [u, v] \in Z(R)$.

Substituting $w$ by $uv$ in the above Eq.(10) and using the Eq.(10), we get

$$[B_1(u, m), w]D_1(v, m) + w[B_1(u, m), D_1(v, m)] + D_1(w, m)[B_1(u, m), v] + [B_1(u, m), D_1(v, m)]v - \quad [u, w]v - w[u, v] \in Z(R),$$

which implies on simplifying

$$[B_1(u, m), w]D_1(v, m) + D_1(w, m)[B_1(u, m), v] \in Z(R)$$

Since $B_1(w, m) \in Z(R)$ by lemma 2.1, the above expression becomes,

$D_1(w, m)[B_1(u, m), v] \in Z(R), since D_1(w, m) \in Z(R)$ by lemma 2.1, hence use remark to conclude $[B_1(u, m), v] \in Z(R)$ and in particular $[B_1(u, m), u] \in Z(R)$. Therefore it is clear from lemma 2.3, $R$ is commutative.

(ii) similar procedure as done above to get the proof.

**Theorem 3.7:** For a prime ring $R$ and a nonzero left ideal $I$ of $R$, suppose that $R$ admits a symmetric biderivations $B_2$ and $D_2$ such that $[w \in Z(R)/B_1(w, m) = D_1(w, m) \neq 0] \neq \emptyset$, then if $R$ satisfies any one of the following conditions:

i. $[B_2(u, m), u] - [u, D_1(u, m)] \in Z(R)$

ii. $[B_2(u, m), u] + [u, D_1(u, m)] \in Z(R)$, for any $u, m \in I$, then $R$ is commutative.

**Proof:** (iii) first assume that $B_2(u, m), u] - [u, D_1(u, m)] \in Z(R)$

If $B_1 = 0$ (or $D_1 = 0$), then $[u, D_1(u, m)] \in Z(R)$. In the case using lemma 2.3, we get the required result.

So we assume that $B_1$ and $D_1$ are nonzero symmetric biderivations, we have $[B_1(u, m), u] - [u, D_1(u, m)] \in Z(R)$, linearizing the above expression, we get

$$[B_1(u + v, m), u + v] - [u + v, D_1(u + v, m)] \in Z(R)$$

Since $B_1(Z(R), m) \neq 0$, there exist $w \in Z(R)$ such that $B_1(w, m) \neq 0$. Replacing $v$ by $vw$ in the above Eq.(11) and using the Eq.(11), we get

$$[B_1(u, m), v] + [B_1(u, m), w]v + w[u, B_1(v, m)] + w[B_1(v, m), v] + [B_1(v, m), u] + [B_1(u, m), u] - \quad [u, D_1(v, m)]v - w[u, v] \in Z(R),$$

which implies on simplifying, we get $B_1(w, m)[v, u] + [B_1(v, m), u] - [u, D_1(v, m)]v - w[u, v] \in Z(R), which is obtained from lemma 2.1, to get

$B_1(w, m)[v, u] - [u, D_1(v, m)]v \in Z(R)$, which implies $B_1(w, m)[v, u] \in Z(R)$ or $D_1(w, m)[v, u] \in Z(R)$. In both the cases using remark, $[u, v] \in Z(R)$ and hence using lemma 2.5(I), we conclude $R$ is commutative.

(ii) similarly we can prove (ii).

**Theorem 3.8:** For a prime ring $R$ and a nonzero left ideal $I$ of $R$, suppose that $R$ admits a symmetric biderivations $B_2$ and $D_2$ such that $[w \in Z(R)/B_1(w, m) = D_1(w, m) \neq 0] \neq \emptyset$, then if $R$ satisfies any one of the following conditions:

i. $B_2(u, m)\{B_2(u, m), v\} \in Z(R)$

ii. $B_2(u, m)\{B_2(u, m), v\} \in Z(R)$, for any $u, m \in I$, then $R$ is commutative.

**Proof:** (i) first assume that $B_2(u, m)\{B_2(u, m), v\} \in Z(R)$, if either of the symmetric biderivations is zero then either $B_2(u, m)\{B_2(u, m), v\} \in Z(R) or uD_2(u, m) \in Z(R)$, hence in both the cases by theorem 3.1, we obtain our result.

Now assume for nonzero symmetric biderivations $B_2$ and $D_1$ such that $B_2(u, m)\{B_2(u, m), v\} \in Z(R)$. Linearizing the above equation, we find that $B_2(u, m)\{B_2(v, m), v\} - uD_2(v, m) - uD_2(u, m) \in Z(R)$.
Since $B_1(Z(R), m) \neq 0$, there exist $w \in Z(R)$ such that $B_1(w, m) \neq 0$, in the same manner $D_1(w, m) \neq 0$. Replacing $v$ by $wv$ in the above Eq.(12) and using the Eq.(12), we get

$$B_1(w, m)(wv) - [B_1(w, m), uv] = D_1(w, m)(wv) - \{w, D_1(w, m)\}v \in Z(R).$$

Use this relation $B_1(w, m) = D_1(w, m) \in Z(R)$ which is obtained from lemma 2.1, to get $B_1(w, m)(wv) - D_1(w, m)(wv) \in Z(R)$, which implies $B_1(w, m)(wv) \in Z(R)$, or $D_1(w, m)(wv) \in Z(R)$. In both the cases using remark, $(wv) \in Z(R)$ and hence using lemma 2.5(ii), we conclude that $R$ is commutative.

(ii) similarly we can prove (ii).

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